3-fold singularity and 5d $\mathcal{N} = 1$ SCFT

Dan Xie

Center for Mathematical Sciences and Applications,
Harvard University

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Motivation

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- Understand properties of those SCFTs such as the moduli space of vacua, global symmetry, duality, etc. Many of those properties are manifest from the singularity point of view.

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- SUSY preserving deformations are classified: the only relevant deformation is the mass deformation which is associated with the operator D_2, and there is no exact marginal deformations. Under certain mass deformation, the IR theory might admit a non-abelian gauge theory description.

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- The Coulomb branch. Notice that there is no R symmetry acting on this branch and therefore we can not parameterize this branch using protected operators. At generic point, the IR theory is an abelian gauge theory plus possibly a **terminal** theory, and the interesting question is to determine the prepotential which is just cubic in the IR gauge fields. There are various type of massive BPS states: electric charged particles, instaton particles, tensile strings, etc.

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- One can also have mixed branch.

Constructing models

There are three important ways of engineering 5d $\mathcal{N} = 1$ SCFT:

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- M theory on 3-fold singularities. Here 3-fold singularity is formed by contracting a divisor: We get massless particles from M2 brane wrapping vanishing 2 cycles inside the divisor, and massless strings from M5 brane wrapping on vanishing 4-cycles. It is argued that the limit is an interacting local quantum field theory (Witten 96, Seiberg 96, Seiberg, Morrison, 96).

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Now for 5d $\mathcal{N} = 1$ SCFT engineered using M theory on 3-fold singularities, we need to consider **resolution** to describe the Coulomb branch!

Our philosophy is that a singularity is the definition of a SCFT, and the first question we would like to answer is: what kind of singularity is needed to define a 5d $\mathcal{N} = 1$ SCFT?

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- 4d $\mathcal{N}=2$ SCFT is also classified by 3 dimensional **canonical** singularity (DX, ST Yau 15)!

We now conjecture that: Every 3 dimensional **canonical** singularity defines a 5d $\mathcal{N}=1$ SCFT!

Definition

A canonical singularity X is defined as follows (Reid 81):

- The Weyl divisor K_X is Q-Cartier, i.e. there is an integer r such that rK_X is a Cartier divisor.
- For any resolution of singularity $f : Y \to X$, with exceptional divisors $E_i \in Y$, we have

$$K_Y = f^* K_X + \sum_i a_i E_i, \tag{2}$$

  with $a_i \geq 0$.

r is called index of the singularity. If $a_i > 0$ for all exceptional divisors, it is called **terminal** singularity.

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- The terminal singularity is classified by following isolated cDV singularity:

$$f(x, y, z) + tg(x, y, z, t) = 0. \tag{4}$$

  here $f(x, y, z)$ is the 2-dimensional ADE singularity.

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- Toric Gorenstein singularity.
- Quasi-homogeneous Hypersurface singularity $f(z_1, z_2, z_3, z_4)$ satisfying the condition

$$f(\lambda^{q_i} z_i) = \lambda f(z_i), \quad \sum q_i > 1. \tag{5}$$

Toric Gorenstein singularity

Let's fix a 3 dimensional lattice $N = Z^3$ and its dual lattice M. 3-d Toric singularity is defined by a convex cone σ which is generated by finite number of lattice vectors $< v_1, \ldots, v_s >$ such that

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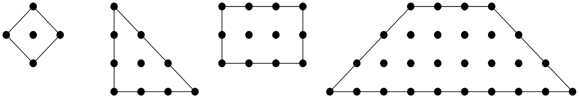
For Gorenstein Toric singularity, one can find a coordinate system such that all the generators are on $z = 1$ hyperplane. The cone cut out a two dimensional convex polygon:



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- They are all canonical singularities.
- The Q factorial terminal singularity is smooth point.

Singular locus and global symmetry

A cone is smooth if the generators form a part of Z basis of N. An affine toric variety is singular, and the singular locus can be easily found from the polygon.

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- The rank of the class group is $d-3$, where d is the number of lattice points on the boundary.

Crepant resolution and Coulomb branch

The crpepant resolution is achieved by finding the unimodular lattice triangulation of the polygon (triangles with only three vertices and no lattice points inside or on the boundary). So the dimension of the Coulomb branch is equal to the number of internal lattice points of the polygon.

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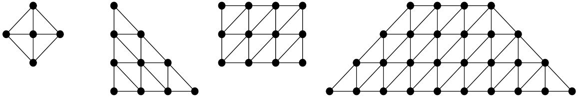
- Many different resolutions, but the resolution is not unique and they are related by the flop: many chambers.
- Each internal vertex in the polygon gives us an exceptional divisor. The number of exceptional divisor is unique and is independent of the crepant resolution.



Dual graph and (p, q) brane web

Let's take a crepant resolution of the convex polygon, and draw the dual diagram: draw a vertex inside triangle and three legs perpendicular to the edges, and we get a web which is nothing but the (p, q) brane web. We have the following simple map:

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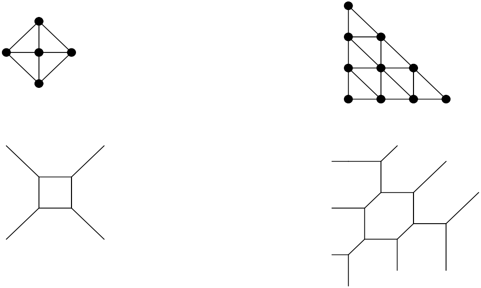
- The edge of (p, q) web gives a curve: we can get string ending on two edges of (p,q) web, and this gives the BPS particles: W bosons, instanton particles, etc.
- The number of semi-infinite edges minus three is the number of mass parameters.



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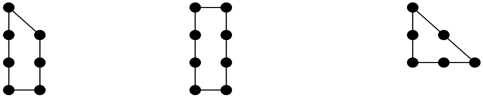

These are expected to be the bi-fundamentals of $SU(n_1) \times SU(n_2) \times U(1)$. The last one has flavor symmetry $SU(2) \times SU(2) \times SU(2)$ and is expected to represent $SU(2)$ tri-fundamental.

Partial resolution and non-abelian gauge theory

We can do the partial resolution by doing a refinement of the convex polygon: We can divide the polygon into several small polygons. The gluing process can be interpreted as gauging non-abelian flavor symmetry of various small polygons:

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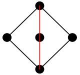
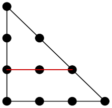
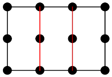


SU(2)



SU(2)–[5]



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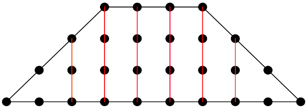


[2]–SU(2)–SU(3)–SU(3)–SU(3)–SU(3)–SU(2)–[2]

Enhanced flavor symmetry from gauge theory

There is a nice formula for the enhanced flavor symmetry (Tachikawa 2015, Yonekura 2015) : for a linear quiver with $SU(N)$ gauge group, prepare two additional Dynkin diagram Γ^+ and Γ^- with the same shape as the quiver gauge theory; Next color a node i in Γ^+ if the following conditions are satisfied:

$$2k_i = 2N_c - N_f. \tag{7}$$

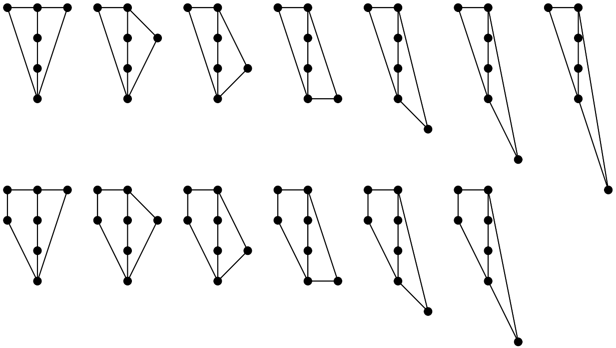
Here k_i is the Chern-Simons level, and N_f is the fundamental flavor. Similarly, color a node in Γ^- if the following conditions are satisfied

$$-2k_i = 2N_c - N_f. \tag{8}$$

The final flavor symmetry can be read from the above two colored Dynkin diagrams (replace $SU(2) - 3$ by $SU(1) - SU(2) - SU(3)$).

Fun with building gauge theories

Now we could play with those polygons and build many interesting theories:



Duality

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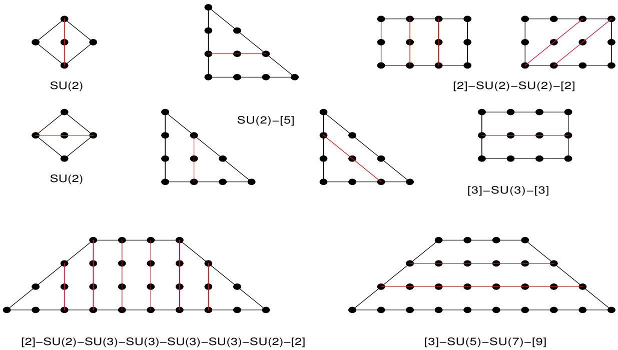

Many interesting new dualities.

Higgs branch

The Higgs branch is related to deformation theory of toric singularity. For isolated singularities, different branches corresponding to different Minkowski summand of the same lattice polygon.

If we compactify 5d theory on a circle, we get a 4d $\mathcal{N} = 2$ theory in the IR. An important question is to write down the Seiberg-Witten curve of these theories. Again, there is an easy way to write down those curves using the mirror symmetry (Hori, Vafa, 2002). The rule is actually the same as the one given in (Aharony, Hanany, Kol, 97).

Other singularities

We can extend the computation to the quotient singularities and the hypersurface singularities: we can compute the dimension of Coulomb branch and the dimension of mass parameters, flavor symmetry, etc. These give many new 5d $\mathcal{N}=1$ SCFTs. Some properties of field theory can be found as follows:

- The Coulomb branch is described by crepant resolution, and the Higgs branch is described by deformation.
- The number of mass parameters is equal to the rank of local class group.
- The dimension of Coulomb branch can be computed by counting the number of crepant divisors.
- The flavor symmetry can be found by looking at one dimensional singular locus and its singular type.

Singularities give us many new interesting $\mathcal{N} = 1$ SCFT. Those theories deserve further study. Here are some interesting further questions

- We found that the ending point of resolution is not often the smooth point, but some Q-factorial terminal singularities. It is interesting to study further those interacting SCFT without the Coulomb branch.
- Further study of duality, i.e. checking duality using superconformal index (HC Kim, S.S Kim, Kimyeong Lee 12).
- BPS spectrum.
- Connection to 4d $\mathcal{N} = 2$ SCFT.
- ...